AD-A040 748 NAVAL RESEARCH LAB WASHINGTON D C F/G 18/12
SHEAR STABILIZATION OF DRIFT WAVES IN NONCIRCULAR CROSS SECTION--ETC(U)
MAY 77 E OTT, W M MANHEIMER, K R CHU
UNCLASSIFIED NRL-MR-3516 NL

| OF |
ADA040748

END
DATE
FILMED
7 - 77

(12)
B.S.

NRL Memorandum Report 3516

Shear Stabilization of Drift Waves in Noncircular Cross Section Axisymmetric Configurations

EDWARD OTT
Department of Electrical Engineering
Cornell University, Ithaca, NY 14853

WALLACE M. MANHEIMER
Plasma Dynamics Branch
Plasma Physics Division

K. R. CHU
Science Applications, Incorporated
McLean, Virginia 22101

May 1977

This report was sponsored by ERDA under Project AEC-AT(49-20),
titled Controlled Thermonuclear Reaction Theory.

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NRL Memorandum Report 3516	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SHEAR STABILIZATION OF DRIFT WAVES IN NONCIRCULAR CROSS SECTION AXISYMMETRIC CONFIGURATIONS		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing NRL problem. 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Edward Ott, Cornell University, Wallace M. Manheimer and K.R. Chu, SAI		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D.C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Problem H02-37 ERDA Contract AEC-AT(49-20)
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE May 1977 13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 21p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. (14) NRL-MR-3516

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This report was sponsored by ERDA under Project AEC-AT(49-20), titled Controlled Thermonuclear Reaction Theory.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Tokamak
Noncircular tokamak
Drift waves
Shear stabilization

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A calculation is given of the shear induced damping of drift waves for noncircular cross section axisymmetric devices in the electrostatic approximation. Comparison of damping rates for circular and elliptical cross section tokamaks shows only a weak dependence on the ellipticity.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

251 950

CONTENTS

SHEAR STABILIZATION OF DRIFT WAVES IN NONCIRCULAR CROSS SECTION AXISYMMETRIC CONFIGURATIONS

I. INTRODUCTION

The role of shear in the stabilization of drift waves has received a great deal of attention [1-11]. (For example, the treatment of the dissipative trapped electron mode instability has recently [9] been studied because of its interest for tokamak transport.) In the most simplified approaches the growth rate may be expressed as

$$\gamma = \gamma_\ell - \gamma_s$$

where γ_ℓ is a mode growth term usually due to an electron dissipation mechanism (e.g., scattering of trapped electrons for the dissipative trapped electron mode, electron Landau resonance for the universal instability[1], finite electron thermal conductivity for the drift dissipative mode), and γ_s is a shear induced damping rate. Shear stabilization results when $\gamma_s > \gamma_\ell$. The calculation of γ_s was given by Pearlstein and Berk[1] for slab geometry $\left[\underset{\sim}{B} = B_0 (\underset{\sim}{z}_0 + \underset{\sim}{y}_0 \, x/L_s)\right.$ where L_s is the "shear length"$\left.\right]$ in their study of the universal instability. They took the electrostatic potential in the form $f(x) \exp(ik_y y)$ and found a mode localized about $x=0$, which propagates energy outward in x until, as a result of the shear, $\omega/v_i \sim |k_\parallel| = k_y |x| / L_s$, where the wave energy is damped via ion Landau resonance ($v_i =$ the ion thermal velocity). This calculation is trivially extended to a circular plasma column equilibrium which models a large aspect ratio tokamak of major radius R if we impose mode periodicity; along the column length, $z=0$ to $z=2\pi R$. The reason for the essential similarity between the slab and circular column geometries is that they both have two directions of uniformity, y and z

Note: Manuscript submitted May 2, 1977.

in the slab model, and Θ and z in the circular cross section case. On the other hand, there is much interest in tokamaks with vertically elongated cross sections due to their potential advantage with respect to achieving higher β and higher current densities[12]. In addition, multipole devices also exhibit magnetic surfaces with noncircular cross section. In these devices the geometrical Θ symmetry of circular cross section is lost.

It is the purpose of this paper to obtain the shear induced damping rate of drift waves in such noncircular cross section devices. We make the following simplifying assumptions: (i) large aspect ratio, (ii) electrostatic perturbation, (iii) modal wavelengths larger than ρ_i(ρ_i is the ion Larmor radius), and (iv) a longitudinal magnetic field much larger than the poloidal field.

Section II describes the geometry and obtains the fundamental two dimensional partial differential equation which governs this problem. Section III presents the solution of the fundamental equation for the shear induced damping under the assumption that mode rational surfaces are decoupled and discusses this assumption. As an example, these results are applied to a particular tokamak equilibrium in Section IV, and it is shown that the shear induced damping rate is very insensitive to the amount of vertical elongation (at least for the example given).

Related work has been reported by Connor and Hastie[7] who use a different method and do not include plasma current (applicable to levitrons and multipoles but not tokamaks). The main difference with their result is that the $d\alpha/d\psi$ terms in the frequency shift d, given in our Eq. (14), would be absent since without plasma current α is constant. Talor[8] has treated a model equation to determine the effect of strong coupling between mode rational surfaces. Here we only treat situations where mode rational surfaces are decoupled by ion Landau damping (cf. discussion at the end of Section III).

II. GEOMETRY AND FUNDAMENTAL EQUATION

We assume a noncircular cylinder of length $2\pi R$ with periodic boundary conditions applied at the column ends (cf. Fig. 1(a)) to simulate a large aspect ratio axisymmetric device. The coordinates are (ψ,χ,z) and are illustrated in Fig. 1, where constant ψ corresponds to a magnetic surface $d\psi = B_\perp dl_\psi$, χ specifies the crossectional position of a point on a flux surface $d\chi=\alpha B_\perp dl_\chi$,z is the length along the cylinder, $B_\perp$ is the poloidal component of $\underset{\sim}{B}$ (component $\perp$ to z), and dl_ψ and dl_χ are differential lengths in the ψ and χ directions. The function α is chosen to make the (ψ,χ) coordinates orthogonal,

$$\nabla \times \alpha \underset{\sim}{B_\perp} = 0. \tag{1}$$

In order to see that this is so assume ψ and χ are orthogonal as in Figure 1 (b) and take the line integral $\oint \alpha \underset{\sim}{B_\perp} \cdot \underset{\sim}{dl}$ around the closed path (a,b,c,d) indicated in Figure 1(b). We see that for an orthogonal coordinate system $\oint \alpha \underset{\sim}{B_\perp} \cdot \underset{\sim}{dl} = 0$ and so (1) follows.

We now set down the basic equations describing the system. In doing this we make the following assumptions to simplify the analysis: (i) $\omega \gg k_\parallel v_i$, (ii) $T_e \gg T_i$, (iii) $\omega \ll k_\parallel v_e$, (iv) $k_\perp^2 \rho_i^2 \ll 1$.

$$\tilde{n}_i = \tilde{n}_e = \tilde{n}, \tag{2}$$

$$\tilde{n}_e = N(\psi)\, e\Phi/T_e, \tag{3}$$

$$-i\omega \tilde{n}_i + \nabla \cdot (\tilde{n}_i \underset{\sim}{v}) = 0, \tag{4}$$

$$\underset{\sim}{v} = \underset{\sim}{v_\perp} + \underset{\sim}{v_\parallel} \tag{5}$$

$$\underset{\sim}{v_\perp} = -B^{-2} \nabla\Phi \times \underset{\sim}{B} + i\omega (B\omega_{ci})^{-1} \nabla_\perp \Phi \tag{6}$$

$$-i\omega v_{\parallel} = -\omega_{ci}^{-1}\, B \cdot \nabla\Phi \qquad (7)$$

where $\omega_{ci} = eB/M_i$ and $E = -\nabla\Phi$.

Eliminating $\tilde{n}$ and v and using tokamak ordering $(B_\perp/B) \ll 1$ and taking $\Phi \sim \exp(ikz)$, $k=n/R$, we obtain a single second order partial differential equation for Φ in terms of the coordinates ψ and χ. The details leading to this equation appear in Appendix A. The result is

$$\left[\omega - \frac{icT_e}{eB} \alpha B_\perp^2 \frac{\partial \ln N}{\partial\psi} \frac{\partial}{\partial\chi} + \frac{c_s^2}{\omega B^2}\left(\alpha B_\perp^2 \frac{\partial}{\partial\chi} + ik\right)^2 - \frac{cT_e}{eB}\frac{\omega}{\omega_{ci}} \alpha B_\perp^2 \left(\frac{\partial}{\partial\psi}\frac{1}{\alpha}\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\chi}\alpha\frac{\partial}{\partial\chi}\right)\right]\varphi = 0, \qquad (8)$$

where $\varphi = e\Phi/T_e$ and $c_s = (T_e/M_i)^{1/2}$. Equation (8) is our basic equation to be studied in detail in the next section. The first term in (8) is due to $\tilde{n}_e$; the second term is due to the ion density perturbation produced by the $E \times B$ drift and background density gradient; the third term is a result of finite ion inertia along field lines; and the last two terms in (8) represent finite ion inertia across B (the polarization drift).

If the destabilizing sources of electron dissipation were included Eq. (3) would take on the general form $\tilde{n}_e = N(\psi)\left[1+i\hat{M}\right](e\Phi/T_e)$ where $\hat{M}$ can be an operator. Different forms of $\hat{M}$ result from different electron dissipation mechanisms (as discussed in Section I). Here we set $\hat{M} \equiv 0$ (so that no instability mechanism is present) and solve for the mode structure and shear induced damping rate. The effect of $\hat{M}$ for specific instabilities can then be included as a perturbation on this basic solution as demonstrated in References 4 and 5.

III. ANALYSIS OF BASIC EQUATION

To analyze Eq. (8) we expand φ in a set of orthogonal basis functions $u_m(\chi,\psi)$

$$\varphi(\chi,\psi) = \sum_m u_m(\chi,\psi)\,\varphi_m(\psi). \tag{9}$$

It is possible to choose the u_m so that the first three terms in (8), which are larger than the last two, are diagonal for all values of ψ. This will be the most convenient choice for the u_m. We choose ($\eta \equiv \alpha B_\perp^2$)

$$u_m = \exp\left[-i\,\frac{m}{nq(\psi)}kB\int^\chi \eta^{-1}\,d\chi\right], \tag{10}$$

so that

$$\oint u_m u_p^* \, d\chi/\eta = \delta_{pm}\oint d\chi/\eta, \tag{11}$$

where δ_{pm} is the Kronecker delta, $*$ denotes the complex conjugate, and the safety factor is $q(\psi) = (2\pi n)^{-1}kB\oint\eta^{-1}d\chi$. Note that the $u_m(\chi,\psi)$ also satisfy the periodicity condition $u_m(\chi,\psi)=u_m(\chi+\oint d\chi,\psi)$ on all magnetic flux surfaces. If we represent Eq. (8) symbolically as $L\varphi=0$ then use of (9) converts (8) to a matrix operator equation

$$\sum_p L_{pm}\varphi_m(\psi) = 0, \tag{10}$$

where each of the L_{pm} is a second order ordinary differential operators in ψ gotten from the relation

$$L_{pm}\varphi_m = \left[\oint dx\,(\alpha B_\perp^2)^{-1}\,u_p^*\,L(u_m\varphi_m)\right]\left[\oint dx(\alpha B_\perp^2)^{-1}\right]^{-1}. \tag{11}$$

From (11) and (8) we obtain

$$L_{pm} = D_{pm} + N_{pm} + P_{pm} \tag{12}$$

where

$$D_{pm} = \delta_{pm}\left\{\omega - \frac{m/n}{q(\psi)}\,\omega_{*e} - \frac{k^2c_s{}^2}{\omega q^2(\psi)}\left[q(\psi) - m/n\right]^2\right\},$$

$$N_{pm} = \frac{cT_e}{eB}\;\frac{\omega}{\omega_{ci}}\left(\frac{kB}{q(\psi)}\right)^2\left(\frac{m}{n}\right)\left(\frac{p}{n}\right)\left[\oint\alpha\eta^{-2}\exp i(F_p - F_m)dx\right]\left[\oint\eta^{-1}dx\right]^{-1},$$

$$P_{pm} = \frac{-cT_e}{eB}\;\frac{\omega}{\omega_{ci}}\left[\oint\eta^{-1}dx\right]^{-1}\oint\alpha^{-1}\exp i\,(F_p - F_m)\left\{\frac{d^2}{d\psi^2}\right.$$

$$\left.-\left[\frac{1}{\alpha}\;\frac{\partial\alpha}{\partial\psi} + 2\,i\,\frac{\partial F_m}{\partial\psi}\right]\frac{d}{d\psi} - \left[\left(\frac{\partial F_m}{\partial\psi}\right)^2 + i\alpha\frac{\partial}{\partial\psi}\left(\alpha^{-1}\,\frac{\partial F_m}{\partial\psi}\right)\right]\right\}dx,$$

where $F_m = kBm/(nq(\psi))\int^x \eta^{-1}\,dx$,

$\omega_{*e} = -ckT_e e^{-1}\partial\ell n\, N/\partial\psi$, and D_{pm}, N_{pm} and P_{pm} represent respectively the first three terms of (8), the fourth term in (8), and the last term in (8). Note that D_{pm} is diagonal while N_{pm} and P_{pm} are non-diagonal, and P_{pm} is a second order differential operator in ψ . Here we only consider the case where it is valid to consider the matrix operator as diagonal. We discuss the conditions under which this is valid later. Ignoring the off diagonal components of L_{pm} we have $L_{mm}\varphi_m=0$ which using the approximation $q(\psi) \cong m/n + q'(\psi_o)\,(\psi - \psi_o)$ ($q' \equiv dq/d\psi$ and $q(\psi_o) \equiv m/n$) may be written as

$$\left\{ \frac{d^2}{d\psi^2} - \frac{\omega_{ci}^2}{c_s^2} \langle B_\perp^2 \rangle^{-1} \left[1 - \frac{\omega_{*e}}{\omega} + d - \left(\frac{kc_s}{\omega}\right)^2 \left(\frac{q'}{q}\right)^2_{\psi=\psi_0} (\psi - \psi_0)^2 \right] \right\} \bar{\varphi}_m = 0 \quad (13)$$

where d is a frequency shift term due to the finite cross-field ion inertia terms

$$d = \frac{c_s^2}{\omega_{ci}^2} \left\{ k^2 B^2 \langle B_\perp^{-2} \rangle + \langle B_\perp^2 \rangle^{-1} \left[\langle B_\perp^2 \left(\frac{\partial F_m}{\partial \psi}\right)^2 \rangle \langle B_\perp^2 \rangle \right.\right.$$

$$\left.\left. - \langle B_\perp^2 \left(\frac{\partial F_m}{\partial \psi}\right) \rangle^2 - \frac{1}{4} \left(\langle B_\perp^2 \frac{d\ell n\alpha}{d\psi} \rangle \right)^2 + \frac{1}{2} \langle B_\perp^2 \rangle \langle \alpha B_\perp^2 \frac{d^2 \alpha^{-1}}{d\psi^2} \rangle \right] \right\} , \quad (14)$$

the flux surface average of a function G is

$$\langle G \rangle = \left[\oint G \, B_\perp^{-1} dl_\chi / \oint B_\perp^{-1} \, dl_\chi \right]_{\psi=\psi_0} ,$$

and the new dependent variable $\bar{\varphi}_m$ has been introduced to eliminate the first derivative term in P_{mm},

$$\bar{\varphi}_m = \exp \left(-\int^{\psi} g d\psi \right) \varphi_m(\psi) ,$$

$$g(\psi) = \left(\oint \alpha^{-1} d\chi\right)^{-1} \left(-\frac{1}{2} \frac{d}{d\psi} \oint \alpha^{-1} d\chi + i \oint \alpha^{-1} \frac{\partial F_m}{\partial \psi} \, d\chi \right).$$

Equation (13) is analogous to the equation of Pearlstein and Berk[1]. Looking for solutions satisfying an outgoing wave boundary conditions we obtain the eigenfunction and wave eigenfrequency,

$$\bar{\varphi}_m(\psi) = \exp\left|-\sigma(\psi-\psi_0)^2/2\right| H_n\left|\sigma^{1/2}(\psi-\psi_0)\right|$$

$$\sigma = i\,\frac{\omega_{ci}}{c_s}\,\langle B_\perp^2\rangle^{-\frac{1}{2}}\,\frac{kc_s}{\omega}\,\frac{q'}{q}\,,$$

$$\omega = \frac{\omega_{*e}}{1+d}\left[1 - i\frac{k^2 c_s^{\,2}}{\omega_{*e}\,\omega_{ci}}\,\langle B_\perp^2\rangle^{1/2}\,\frac{q'}{q}\,(2n+1)\right], \qquad (15)$$

where H_n are Hermite polynomials. For d<<1 we may approximate the shear induced damping rate for the least damped mode (n=0) as

$$\gamma_{sh} \cong -\,k^2 c_s^{\,2}\,\omega_{ci}^{\,-1}\,\langle B_\perp^2\rangle^{1/2}\,q'(\psi)/q(\psi) \qquad (16)$$

which will be evaluated in the next section for a specific equilibrium configuration.

We now discuss the conditions under which the diagonal approximation to (10) is valid.

First of all, if the plasma has a circular cross section, α,η and F_m are all independent of χ, so that the matrix is diagonal because of the orthogonality relation. For noncircular cross section, the matrix elements clearly approach zero as m-p approaches infinity. The reason is that the matrix element is the integral of a smoothely varying function (the poloidal field) times a very rapidly oscillating function.

In order to truncate the matrix at a one by one, we must invoke ion Landau damping, which is not explicitly included in Eq. (8). As the wave propagates away from the mode rational surface, the value of $k_\parallel$ increases. When the distance from the rational surface is large enough that $\omega/k_\parallel v_i$

approaches about 2 or 3, the ion Landau damping gets very strong. The energy propagating away from the mode rational surface is absorbed in this region and the eigenfunction decays sharply in space. If the spatial point at which $\omega/k_\parallel v_i \sim 3$ is less than halfway between the neighboring rational surfaces, solutions to the diagonal matrix (including ion Landau damping) are spatially isolated from one another. Thus the eigenfunctions of the diagonalized matrix are a good approximation to the solution of Eq. (10).

The distance between rational surfaces, $\delta\psi$, is

$$\delta\psi = (n dq/d\psi)^{-1} .$$

The distance at which $\omega/k_\parallel v_i \simeq 3$, $\Delta\psi$, is given by

$$\Delta\psi = \frac{\omega q R}{3 n v_i dq/d\psi} .$$

where we have used $k = n/R$. Thus the condition that the matrix can be diagonal reduces to $\delta\psi > 2\Delta\psi$ or

$$\frac{2\omega R q}{3 v_i} < 1. \qquad (17)$$

The work of Taylor[8] applies to the limit opposite to (17).

IV. EVALUATION OF SHEAR STABILIZATION FOR A SPECIFIC EQUILIBRIUM

We consider an equilibrium which we partially specify by choosing the axial current density as a function of ψ,

$$J_z(\psi) = J_0 \text{ for } 0 \leq \psi < \psi_0 , \tag{18a}$$

$$J_z(\psi) = 0 \text{ for } \psi > \psi_0 . \tag{18b}$$

Thus J_z is a constant inside the flux surface $\psi = \psi_0$ and is zero outside this flux surface. Note that the plasma will be assumed to extend into the current free region, $\psi > \psi_0$. We take the surface $\psi = \psi_0$ to be an ellipse of ellipticity κ so that the following equilibrium solution appies in $\psi < \psi_0$ (cf. Ref. 12),

$$\psi = \frac{\kappa B}{2qR} \rho^2, \; \rho^2 = x^2 + y^2/\kappa^2, \tag{19a}$$

$$B_\perp = |\nabla\psi| = \kappa B \, (x^2 + y^2\kappa^{-4})^{1/2} \, (qR)^{-1} . \tag{19b}$$

This equilibrium has all magnetic surfaces elliptical with the same ellipticity. Also, since J_z is constant in $\psi < \psi_0$, $q(\psi)$ is also constant in $\psi < \psi_0$, and there is no shear. Outside the $\psi = \psi_0$ surface $q(\psi)$ is not constant, and there is shear. We evaluate the shear induced damping in $\psi > \psi_0$ for a point close to the $\psi = \psi_0$ surface. The geometry dependent factor in (16) is

$$\langle B_\perp^2 \rangle^{1/2} (q'/q) = \left(\frac{\oint B_\perp dl_\chi}{\oint B_\perp^{-1} dl_\chi} \right)^{1/2} \frac{\frac{d}{d\psi} \oint (\alpha B_\perp)^{-1} d\chi}{\oint (\alpha B_\perp)^{-1} d\chi} . \tag{20}$$

The evaluation is performed in Appendix B for the particular equilibrium under consideration and we obtain from (16)

$$\gamma_{sh} = -2\frac{kc_s^2}{\omega_{ci}\rho} f(\kappa), \; f(\kappa) = \kappa^{-2}\left|\frac{1}{2}(\kappa^2 + 1)\right|^{3/2}. \qquad (21)$$

The variation of γ_{sh} with κ[as given by the function $f(\kappa)$] is very weak, as can be seen by the following numbers: f(1) = 1 (circular), f (1.5) = 0.988, f(2.5) = 1.104, f(3) = 1.242, f(3.5) = 1.392. Thus for κ between 1 and 2.5 the value of γ_{sh} is within 10% of its value for the circular case. In particular, from a local theory calculation of the dissipative trapped electron mode stability in vertically elongated tokamaks[13] it was found that the growth rate was reduced by elongation. Since the shear damping appears to be approximately independent of elongation (for this example), this indicates the possibility of complete stabilization by vertical elongation.

This work was supported by the United States Energy Research and Development Administration.

REFERENCES

1. L. D. Pearlstein and H. L. Berk, Phys. Rev. Lett. 23, 220 (1969).
2. P. H. Rutherford and E. A. Frieman, Phys. Fluids 10, 1007 (1967).
3. N. A. Krall and M. N. Rosenbluth, Phys. Fluids 8, 1468 (1965).
4. C. S. Liu, M. N. Rosenbluth and C. W. Horton, Jr., Phys. Rev. Lett. 29, 1489 (1972).
5. N. T. Gladd and W. Horton, Jr., Phys. Fluids 16, 879 (1973).
6. B. Coppi, G. Laval, R. Pellat and M. N. Rosenbluth, Plasma Phys. 10, 1 (1968).
7. J. W. Connor and R. J. Hastie, Plasma Phys. 17, 109 (1975).
8. J. B. Taylor, in Plasma Physics and Controlled Nuclear Fusion Research 1976 (to be published).
9. R. Z. Sagdeev and A. A. Galeev, Comments on Plasma Phys. and Controlled Fusion 1, 23 (1970).
10. W. Horton et al., in Plasma Physics and Controlled Nuclear Fusion Research (IAEA, Vienna, 1975), Vol. 1, p. 541.
11. W. M. Manheimer, Phys. Fluids 19, 335 (1976).
12. A. H. Glasser, E. A. Freiman, and S. Yoshikawa, Phys. Fluids 17, 181 (1974).
13. E. Ott, K. R. Chu and W. M. Manheimer, NRL Memorandum Report 3459.

APPENDIX A

DERIVATION OF EQ. (8)

Equation (8) follows simply from Eqs. (2) - (7). Equations (2) and (3) express n_i in terms of Φ, while Eqs. (6) and (7) express v_i in terms of Φ. Inserting these expressions in the ion continuity equation then gives a single equation for Φ. The only problem is to express gradients, divergences, and unit vectors in fluc co-ordinates. The length element is

$$d\underset{\sim}{s} = \frac{1}{B_\perp} d\underset{\sim}{\psi} + \frac{1}{\alpha B_\perp} d\underset{\sim}{\chi} + d\underset{\sim}{z}. \tag{A1}$$

The unit vecotrs perpendicular and parallel to the magnetic field are then

$$\underset{\sim}{i}_1 \equiv \underset{\sim}{i}_\psi$$

$$\underset{\sim}{i}_2 = \frac{B_\perp \underset{\sim}{i}_z - B_z \underset{\sim}{i}_\psi}{(B_\perp^2 + B_z^2)^{1/2}}$$

$$\underset{\sim}{i}_3 \equiv \underset{\sim}{i}_\parallel = \frac{B_\perp \underset{\sim}{i}_\psi + B_z \underset{\sim}{i}_z}{(B_\perp^2 + B_z^2)^{1/2}}$$

The components of the gradient perpendicular and parallel to the magnetic field are

$$\underset{\sim}{\nabla}_\perp = \underset{\sim}{i}_\psi B_\perp \frac{\partial}{\partial\psi} + \underset{\sim}{i}_2 (\underset{\sim}{i}_2 \cdot \underset{\sim}{\nabla})$$

$$= \underset{\sim}{i}_\psi B_\perp \frac{\partial}{\partial\psi} + \underset{\sim}{i}_2 \frac{B_\perp \frac{\partial}{\partial z} - \alpha B_z B_\perp \frac{\partial}{\partial \chi}}{(B_\perp^2 + B_z^2)^{1/2}} \tag{A2}$$

$$\nabla_{\parallel} = \nabla_3 = i_{\parallel} \frac{\alpha B_{\perp}^2 \frac{\partial}{\partial x} + B_z \frac{\partial}{\partial z}}{(B_{\perp}^2 + B_z^2)^{1/2}} \tag{A3}$$

Finally, the divergence of a vector A is

$$\mathrm{div}\, A = \alpha B_{\perp}^2 \left\{ \frac{\partial}{\partial \psi} \frac{A_{\psi}}{\alpha B_{\perp}} + \frac{\partial}{\partial x} \frac{A_x}{B_{\perp}} + \frac{1}{\alpha B_{\perp}^2} \frac{\partial}{\partial z} A_z \right\} . \tag{A4}$$

By using Eqs. (A2) - (A4) and also assuming $B_z \gg B_{\perp}$ (consistent with tokamak ordering), and also $\alpha B_{\perp} \frac{\partial}{\partial x} \gg k_z$ (consistent with propagation nearly perpendicular to B), Eqs.(2) through (7) can be manipulated into Eq. (8).

APPENDIX B

EVALUATION OF δ_{sh} FOR ELLIPTICAL MODEL

Since $J_z=0$ in $\psi>\psi_o$, $\nabla x B_\perp = 0$ there and consequently Eq. (1) implies that α is constant in $\psi > \psi_o$. Thus

$$d/d\psi\left[\oint(\alpha B_\perp^2)^{-1}dx\right] = -2\oint B_\perp^{-3}\,(dB_\perp/dl_\psi)dl_x$$

Also from $\nabla x \underset{\sim}{B}_\perp = 0$ it follows that $B_\perp^{-1}dB_\perp/dl_\psi = -R_c^{-1}$,

where R_c is the radius of curvature of the flux surface in the constant z plane. Thus

$$\langle B_\perp^2\rangle^{1/2}\,(q'/q)=2\left(\frac{\int_o^\rho B_\perp \frac{dl_x}{dx}\,dx}{\int_o^\rho B_\perp^{-1}\frac{dl_x}{dx}\,dx}\right)^{1/2}\frac{\int_o^\rho (B_\perp^2 R_c)^{-1}\frac{dl_x}{dx}\,dx}{\int_o^\rho B_\perp^{-1}\frac{dl_x}{dx}\,dx}\,. \quad \text{(B1)}$$

To evaluate this quantity on a surface of constant ψ we note that near $\psi = \psi_o$ in $\psi > \psi_o$ the flux surface is approximately elliptical and $B_\perp$ is approximately $B_\perp$ on the $\psi = \psi_o$ surface. To evaluate the integrals we need $B_\perp$, R_c and dl_x/dx as functions of x on the magentic surface (ρ = constant). From Eqs. (19) we obtain

$$B_\perp = \frac{B\rho}{q_oR}\left[1 + (\kappa^2-1)\,(x/\rho)^2\right]^{1/2}, \quad \text{(B2)}$$

$$dl_x/dx = \left[1 + (dy/dx)^2\right]^{1/2} = \left[1 + (\kappa^2-1)\,(x/\rho)^2\right]^{1/2}\left[1 - (x/\rho)^2\right]^{-1/2}. \quad \text{(B3)}$$

To find R_c we express $\underset{\sim}{i}_\psi$, the unit normal to the magentic surface, $\underset{\sim}{i}_\psi = \nabla\rho^2/|\nabla\rho^2|$. R_c is then given by

$$R_c^{-1} = |d\underset{\sim}{i}_\psi/dl_\chi| = (\partial \underset{\sim}{i}_\psi/\partial x)_\rho |(dl_\chi/d\chi)^{-1},$$

and thus

$$R_c^{-1} = \kappa\rho^{-1} [1 + (\kappa^2 - 1) (x/\rho)^2]^{-3/2}. \qquad \text{(B4)}$$

Putting (B-2) to (B-4) in (B-1) we obtain

$$\langle B_\perp^2 \rangle^{1/2} (q'/q) = 2 (\kappa^2\rho)^{-1} [\tfrac{1}{2} (\kappa^2 + 1)]^{3/2},$$

from which Eq. (21) follows.

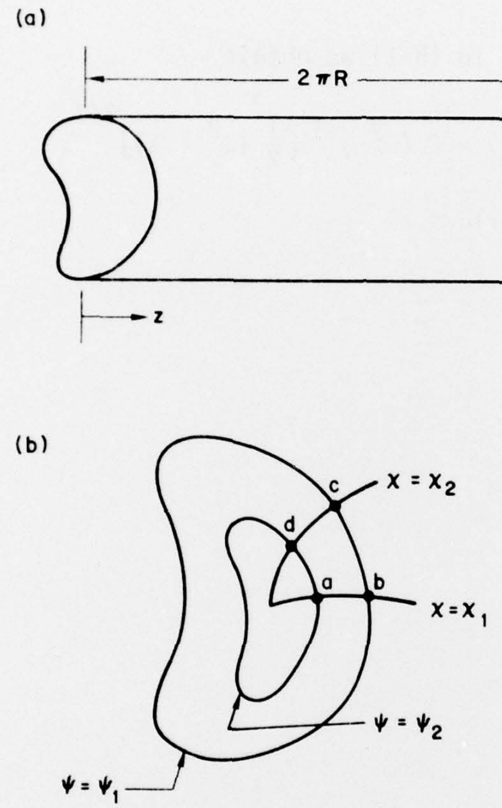

Fig. 1 — (a) Noncircular cylinder, and (b) ψ, χ coordinates in a constant z plane